Proof Cauchy Problem

We need to prove that $\frac{d}{dt}x(t) = Ax(t)$ with initial condition $x(0) = u \in \mathbb{R}^2$

Compute the derivative $\frac{d}{dt}x(t)$:

$$\frac{d}{dt}x(t) = \frac{d}{dt}e^{tA}u = u\frac{d}{dt}e^{tA} = u \cdot A e^{tA} = Ax(t)$$

since the derivative of e^{tA} can be computed as follows:

$$\frac{d}{dt}e^{tA} = \lim_{h\to 0}\frac{e^{(t+h)A} - e^{tA}}{h}$$

$$= \lim_{h\to 0}\frac{e^{tA}e^{hA} - e^{tA}}{h}$$

$$= e^{tA}\lim_{h\to 0}\frac{e^{hA} - I}{h}$$

$$= e^{tA}A$$

$$e^{hA} = \sum_{k=0}^{\infty}\frac{(hA)^k}{k!} \Rightarrow \frac{e^{hA} - I}{h} = \frac{\sum_{k=0}^{\infty}\frac{(hA)^k}{k!} - I}{h}$$

$$= \frac{\sum_{k=1}^{\infty}\frac{(hA)^k}{k!} + I - I}{h}$$

$$= \sum_{k=1}^{\infty}\frac{h^k A^k}{k!}$$

$$= \sum_{k=0}^{\infty}\frac{h^k A^{k+1}}{(k+1)!} =$$

$$= A + \sum_{k=1}^{\infty}\frac{h^k A^{k+1}}{(k+1)!} \xrightarrow{h\to 0} A$$

Type 1

$$x(t) = \begin{pmatrix} e^{t\lambda} & k_1 \\ e^{t\mu} & k_2 \end{pmatrix}$$

proof:

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

Compute e^{tA}

$$e^{tA} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{\begin{pmatrix} t\lambda & 0 \\ 0 & t\mu \end{pmatrix}^k}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{\begin{pmatrix} (t\lambda)^k & 0 \\ 0 & (t\mu)^k \end{pmatrix}}{k!}$$

$$= \sum_{k=0}^{\infty} \begin{pmatrix} \frac{(t\lambda)^k}{k!} & 0 \\ 0 & \frac{(t\mu)^k}{k!} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{k=0}^{\infty} \frac{(t\lambda)^k}{k!} & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{(t\mu)^k}{k!} \end{pmatrix}$$

$$= \begin{pmatrix} e^{t\lambda} & 0 \\ 0 & e^{t\mu} \end{pmatrix}$$

Type 2:

$$x(t) = e^{ta} \begin{pmatrix} k_1 \cos(tb) & -k_2 \sin(tb) \\ k_1 \sin(tb) & k_2 \cos(tb) \end{pmatrix}$$

$$A \begin{matrix} a = r\cos\theta \\ b = r\sin\theta \end{matrix}$$

proof:

We interpret the map T given by the matrix A algebraically by identifying $\mathbb{R}^2$ with the complex space $\mathbb{C}$

$$(x,y) \leftrightarrow x + iy$$

We get the following correspondence for T:

$$\begin{array}{ccc} (x,y) & \longleftrightarrow & x+iy \\ \updownarrow T & & \updownarrow \text{Multiplying by } a+ib \\ (ax-by,\, bx+ay) & \longleftrightarrow & ax-by+i(bx+ay) \end{array}$$

In the same way there is a correspondence e^{a+ib}.

With $e^A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ this results in the following scheme:

$$\begin{array}{ccc} (x,y) & \longleftrightarrow & x+iy \\ \updownarrow e^A & & \updownarrow e^{a+ib} \\ (a_1x+a_2y,\, a_3x+a_4y) & \longleftrightarrow & e^a(x\cos b - y\sin b + i(x\sin b + y\cos b)) \end{array}$$

By comparing the coefficients we can conclude that the matrix e^A can be represented as follows:

$$e^A = e^a \begin{pmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{pmatrix}$$

$$e^{a+ib}(x+iy) = e^a e^{ib}(x+iy)$$

$$= e^a(x(\cos b + i\sin b) + yi(\cos b + i\sin b)) \qquad e^{ib} = \cos b + i\sin b$$

$$= e^a(x\cos b - y\sin b + i(y\sin b + y\cos b))$$

Type 3:

$$x(t) = e^{t\lambda} \begin{pmatrix} k_1 \\ k_1 t + k_2 \end{pmatrix}$$

proof:

The matrix A can be split up in the following way:

$$A = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix} = \lambda \cdot I + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

For the matrix $M = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ the following equation holds which can be easily computed:

$$M^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

Consequently, we get $M^k = 0$ for all $k \geq 2$

Thus, we can compute e^{tA} as follows:

$$\begin{aligned} e^{tA} &= e^{t(\lambda I + M)} \\ &= e^{t\lambda I + \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}} \\ &= e^{t\lambda I} \cdot e^{\begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}} \\ &= e^{t\lambda I} \cdot \left(I + \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix} \right) \quad \text{using the above equation } M^k = 0, k \geq 2 \\ &= e^{t\lambda} \cdot \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \end{aligned}$$